

Reply to the Comment by Zhang and Li [J. Math. Phys. 56, 084101, 2015] on the Paper [J. Math. Phys. 56, 032104, 2015]

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Abstract:

In a recent paper Zhang and Li have doubted our claim that whenever a nonlinear equation has solutions in terms of the Jacobi elliptic functions $\text{cn}(x, m)$ and $\text{dn}(x, m)$, then the same nonlinear equation will necessarily also have solutions in terms of $\text{dn}(x, m) \pm \sqrt{m}\text{cn}(x, m)$. We point out the flaw in their argument and show why our assertion is indeed valid.

In a series of recent papers we had shown through a large number of examples [1, 2, 3] that whenever a nonlinear equation, discrete or continuous, integrable or nonintegrable, coupled or uncoupled, local or nonlocal has periodic solutions in terms of Jacobi elliptic functions $\text{dn}(x, m)$ and $\text{cn}(x, m)$, then the same equation will necessarily have solutions in terms of their linear combinations $\text{dn}(x, m) \pm \sqrt{m}\text{cn}(x, m)$. Recently Zhang and Li [4] have examined our claim in the context of nonlinear Schrödinger equation and have claimed that such an assertion is impossible. The purpose of this reply is to point out a serious flaw in their argument.

Zhang and Li start with the nonlinear Schrödinger equation (NLSE)

$$iu_t + u_{xx} + b|u|^2u = 0. \quad (1)$$

On assuming

$$u(x, t) = e^{i(kx - \omega t)}\phi(\zeta), \quad \zeta = x - vt, \quad (2)$$

they reduce the NLSE to

$$\phi''(\zeta) = a\phi + b\phi^3, \quad (3)$$

where $a = \omega - k^2$, $v = 2k$. Then they went on to obtain various solutions to the Eq. (3) including those in terms of $\text{dn}(x, m)$ and $\text{cn}(x, m)$. Up to this point we completely agree with Zhang and Li [4].

The crucial flaw in their argument came at the stage when they asserted that if the nonlinear Eq. (3) has two solutions $\phi_1(\zeta)$ and $\phi_2(\zeta)$ then $\Phi(\zeta) = \phi_1(\zeta) + \phi_2(\zeta)$ can be a solution of the nonlinear Eq. (3) only if

$$\phi_1^2\phi_2 + \phi_2^2\phi_1 = 0. \quad (4)$$

This assertion is incorrect. In particular, if ϕ_1 and ϕ_2 are such that

$$\phi_2^2 = \phi_1^2 + c, \quad (5)$$

then also $\phi(\zeta)$ can be a solution of Eq. (3), where c is a constant. And this is precisely true in the case of the elliptic functions $\text{dn}(\zeta, m)$ and $\sqrt{m}\text{cn}(\zeta, m)$. In particular, while $\text{dn}(\zeta, m)$ and $\text{cn}(\zeta, m)$ are distinct periodic functions with periods $2K(m)$ and $4K(m)$ (where $K(m)$ is complete elliptic integral of the first kind [5]), they satisfy the identity

$$\text{dn}^2(x, m) = m\text{cn}^2(x, m) + (1 - m). \quad (6)$$

And precisely because of such an identity that our assertion, as has been proved by numerous examples, is indeed correct.

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References

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